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Ward–Takahashi identity at finite temperature

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Abstract. We provide a perturbative check of Ward-Takahashi identities of QED at finite temperature using a new real time formalism, thermo field dynamics. Gauge invariance is shown to hold up to the two-loop level except one special case. In verifying the identities we find a consistent rule in dealing with products of singular terms which are characteristic of the real time formalism.

1. Introduction

The present work is concerned with the validity of Ward–Takahashi (WT) identities at finite temperature in a new real time formalism called thermo field dynamics (TFD).

The WT identity is the expression of the invariance of a given theory under a certain symmetry transformation. Notable examples are the identity due to Ward [1],

$$\partial S_F(k)/\partial k_\mu = iS_F(k)r_\mu S_F(k) \tag{1.1}$$

and Takahashi's identity [2],

$$S_F(k_1) - S_F(k_2) = iS_F(k_1)(k_1 - k_2)S_F(k_2) \tag{1.2}$$

in quantum electrodynamics (QED). At zero temperature ($T=0$) WT identities are derived in the path-integral formalism as follows. One starts with a generating functional

$$W(J) = \int [d\phi] \exp\left(i \int d^4x(\mathscr{L}(x) + J\cdot\phi)\right)$$

where ϕ and J collectively denote the fields in the theory and their sources, and we have omitted the normalisation factor. Under a symmetry transformation $\phi \to \phi + \varepsilon\delta\phi$ (ε: infinitesimal parameter) the generating functional is invariant; this leads to the basic identity

$$0 = \frac{\delta W}{\delta\varepsilon} = \int d[\phi] J\delta\phi \exp\left(i \int d^4x\{\mathscr{L}(x) + J\cdot\phi\}\right). \tag{1.3}$$

Various WT identities are derived by taking functional derivatives $\delta/\delta J(x)$, $\delta/\delta J(g)\dots$ of the basic identity and then setting the sources to be zero.

Since we shall concern ourselves with WT identities at finite temperature ($\equiv T\neq 0$) we will give a brief review of the derivation of WT identities in TFD [3-9, 13]. There are two important features in this new real time formalism. One of them is the presence of the so-called tilde field, denoted by $\tilde{\phi}$, which emerges naturally as a result of the

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analytic continuation in time [6]. The resulting Lagrangian, $\mathscr{L}$, becomes $\mathscr{L}(\phi, \tilde{\phi}) = \mathscr{L}(\phi) - \mathscr{L}(\tilde{\phi})$ [3-4]. Further in the path-integral formulation [7] one must introduce an independent source, $\tilde{J}$, for $\tilde{\phi}$. Thus the generating function is given by

$$W(J, \tilde{J}) = \int [\mathrm{d}\phi\, \mathrm{d}\tilde{\phi}] \exp\left(\mathrm{i} \int \mathrm{d}^4x \{\mathscr{L}(\phi, \tilde{\phi}) + \boldsymbol{J} \cdot \phi + \tilde{\boldsymbol{J}} \cdot \tilde{\phi}\}\right). \tag{1.4}$$

WT identities in TFD [8] are obtained by performing a symmetry transformation and taking functional derivatives $\delta/\delta J$ and $\delta/\delta\tilde{J}$. (Here we note that the transformation of $\tilde{\phi}$ is identical to that of ϕ.) They give in general relations among N-point Green functions such as

$$\langle 0(\beta)| T\phi_1(x)\phi_2(y) \ldots \phi_N(z)|0(\beta)\rangle \tag{1.5}$$

$$\langle 0(\beta)| T\tilde{\phi}_1(x)\tilde{\phi}_2(y) \ldots \tilde{\phi}_N(z)|0(\beta)\rangle \tag{1.6}$$

$$\langle 0(\beta)| T\phi_1(x)\phi_2(y) \ldots \tilde{\phi}_N(z)|0(\beta)\rangle \tag{1.7}$$

etc, where $|0(\beta)\rangle$ is the finite temperature vacuum. As presented above the derivations at $T = 0$ and $T \neq 0$ are similar to each other. The replacements $|0\rangle \to |0(\beta)\rangle$, $\phi \to (\phi, \tilde{\phi})$, $J \to (J, \tilde{J})$ give one WT identity at $T \neq 0$. So far as the physical sector, namely the part consisting solely of ϕ fields, is concerned, WT identities at $T \neq 0$ are identical in appearance with those at $T = 0$ (although one must note that the Green functions are now defined on the finite temperature vacuum $|0(\beta)\rangle$). Therefore they are given in a covariant manner as at $T = 0$.

What we would like to do in this paper is to check perturbatively the validity of a few WT identities at $T \neq 0$. At this point we must refer to another important feature of TFD. Perturbative calculation requires vertices and propagators. The vertices are read off directly from $\mathscr{L}$. For instance, in the case of real scalar theory, we have $\lambda\phi^4$ and $-\lambda\tilde{\phi}^4$. The propagator, called thermo propagator in TFD, was first obtained by Matsumoto [4]. In the thermo propagator one finds a great difference from the propagator in the old real time formalism [11, 12]. The scalar thermo propagator is a 2×2 matrix

$$\Delta(k) = \begin{vmatrix} \langle\phi\phi\rangle & \langle\phi\tilde{\phi}\rangle \\ \langle\tilde{\phi}\phi\rangle & \langle\tilde{\phi}\tilde{\phi}\rangle \end{vmatrix} \equiv \begin{vmatrix} \langle 11\rangle & \langle 12\rangle \\ \langle 21\rangle & \langle 22\rangle \end{vmatrix}$$
$$= \begin{vmatrix} \Delta_0^1(k) + 2\pi\delta(k^2 - m^2)f_{\mathrm{B}}(k) & 2\pi\delta(k^2 - m^2)g_{\mathrm{B}}(k) \\ 2\pi\delta(k^2 - m^2)g_{\mathrm{B}}(k) & \Delta_0^2(k) + 2\pi\delta(k^2 - m^2)f_{\mathrm{B}}(k) \end{vmatrix} \tag{1.8}$$

where

$$\Delta_0^1(k) = \mathrm{i}(k^2 - m^2 + \mathrm{i}\varepsilon)^{-1} \qquad \Delta_0^2(k) = -\mathrm{i}(k^2 - m^2 - \mathrm{i}\varepsilon)^{-1}$$
$$f_{\mathrm{B}}(k) = [\exp(\beta|k_0|) - 1]^{-1} \qquad g_{\mathrm{B}}(k) = \exp(\tfrac{1}{2}\beta|k_0|)[\exp(\beta|k_0|) - 1]^{-1}. \tag{1.9}$$

As is manifest in the structure of the propagator the finite temperature perturbation in TFD destroys Lorentz invariance. Therefore it appears there is no guarantee that finite temperature WT identities formulated in a covariant manner are satisfied at each order of perturbation. In fact one finds a counter example, although very special, where the breaking is precisely due to the loss of covariance in the thermo propagator. Therefore it is desirable to check explicitly if the new perturbation respects WT identities. The check is a non-trivial task. The thermo propagator involves the δ function and thus in higher orders one has to deal with the products of distributions. Depending on how one defines them the answer can differ. In the next section we shall examine the validity of WT identities of QED up to two-loop level.

2. Ward–Takahashi identities of QED in thermo field dynamics

In this section we investigate various identities of QED and investigate their validity at finite temperature perturbatively.

2.1. Takahashi's identity

A finite temperature version of Takahashi's identity reads as

$$S_F(k_1) - S_F(k_2) = iS_F(k_1)(k_1 - k_2)S_F(k_2) \tag{2.1}$$

where

$$S_F(k) = (k+m)\begin{vmatrix} \Delta_0^1(k) - S_\beta(k) & \varepsilon(k_0)S'_\beta(k) \\ \varepsilon(k_0)S'_\beta(k) & \Delta_0^2(k) + S_\beta(k) \end{vmatrix} \tag{2.2}$$

where $\varepsilon(k_0)$ is a step function and

$$S_\beta(k) = \frac{2\pi\delta(k^2 - m^2)}{\exp(\beta|k_0|)+1} \equiv 2\pi\delta(k)f_F(k)$$

$$S'_\beta(k) = 2\pi\delta(k^2 - m^2)\frac{\exp(\frac{1}{2}\beta|k_0|)}{\exp(\beta|k_0|)+1} \equiv 2\pi\delta(k)g_F(k)$$

$$\Delta_0^1(k) = \frac{i}{k^2 - m^2 + i\varepsilon} \qquad \Delta_0^2(k) = \frac{i}{k^2 - m^2 - i\varepsilon}. \tag{2.3}$$

(Note a change in sign in $\Delta_0^2(k)$ relative to the boson case, (1.9). This is due to the anti-commuting nature of fermion as noted in [5].)

If one denotes

$$iS_F(k_1)(k_1 - k_2)S_F(k_2) \equiv \begin{vmatrix} (a) & (b) \\ (c) & (d) \end{vmatrix} \tag{2.4}$$

one finds

$$(a) = -(\not{K}_2 + m)\{\Delta_0^1(k_2) - S_\beta(k_2)\} + (\not{K}_1 + m)\{\Delta_0^1(k_1) - S_\beta(k_1)\} \tag{2.5}$$

$$(b) = -(\not{K}_2 + m)\varepsilon(k_{20})S'_\beta(k_2) + (\not{K}_1 + m)\varepsilon(k_{10})S'_\beta(k_0) = (c) \tag{2.6}$$

$$(d) = -(\not{K}_2 + m)\{\Delta_0^2(k_2) - S_\beta(k_2)\} + (\not{K}_1 + m)\{\Delta_0^2(k_1) - S_\beta(k_1)\}. \tag{2.7}$$

In obtaining (2.5)-(2.7) we have made use of

$$(k^2 - m^2)\delta(k^2 - m^2) = 0. \tag{2.8}$$

The RHS, (a)-(d), coincides with the LHS of (2.1).

We further note that in this case an identity holds for a sub-sector of S_F, i.e.

$$S_F^{(11)}(k_1) - S_F^{(11)}(k_2) = iS_F^{(11)}(k_1)(\not{K}_1 - \not{K}_2)S_F^{(11)}(k_2) \tag{2.9}$$

where $S_F^{(11)}(k)$ denotes the (11)-element of $S_F(k)$. A cautionary remark we like to give here is that Takahashi's identities at finite temperature ((2.1) and (2.9)) hold in a weak sense since we made use of (2.8). A situation arises in higher orders where the term (2.8) goes with an extra factor $1/k^2 - m^2$ and thus cannot be set to be zero. (See the two-loop check of gauge invariance (2.22)-(2.25).) Therefore one must take due care in applying $T \neq 0$ Takahashi's identities.

2.2. Ward's identity

$$\partial S_F(k)/\partial k_\mu = \mathrm{i} S_F(k) r_\mu S_F(k). \tag{2.10}$$

As one can easily verify by inserting S_F of (2.2), Ward's identity holds for $\mu = 1, 2, 3$ but not for $\mu = 0$. The reason is that the statistical factor contains k_0 and thus an extra term is present in the LHS of (2.10). This is a very special type of identity. Identities in general do not involve momentum derivatives. We presented it here only to remind that in a special case an identity may not hold in the same form as in zero temperature. (A further discussion on Ward's identity can be found in [9].)

In the following three examples we shall check the gauge invariance in QED at finite temperature. For simplicity we shall consider only the physical sector.

2.3. Gauge invariance $P_\mu \pi_{\mu\nu}(p) = 0$ (one-loop)

Gauge invariance leads to the well known identity for the vacuum polarisation tensor, $\pi_{\mu\nu}(p)$:

$$P_\mu \pi_{\mu\nu}(p) = 0. \tag{2.11}$$

Figure 1 represents the one-loop diagram (at $T \neq 0$) for the physical sector,

$$P^\mu \pi_{\mu\nu} = -(-\mathrm{i}e)^2 \,\mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} S_F^{(11)}(p+k) \not{P} S_F^{(11)}(k) r_\nu, \tag{2.12}$$

where we define the integral dimensionally to allow the arbitrary change of variable. By the use of Takahashi's identity (2.9), one obtains

$$P^\mu \pi_{\mu\nu} = e^2 \,\mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} \{S_F^{(11)}(p+k) - S_F^{(11)}(k)\} r_\nu \tag{2.13}$$

and by changing the variable $p + k \to k$ in the first term,

$$= 0.$$

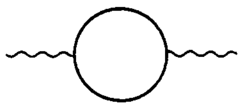

Figure 1. One-loop photon self-energy diagram.

2.4. Photon-photon scattering (figure 2)

A suitable assignment of momenta makes the proof very easy.

I_a(figure 2(a))

$$= \mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} S_F^{(11)}(k) r_\mu S_F^{(11)}(k+p_2) r_\nu S_F^{(11)}(k+p_2+p_3) r_\rho S_F^{(11)}$$
$$\times (k+p_2+p_3+p_4) r_\sigma \tag{2.14}$$

I_b(figure 2(b))

$$= \mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} S_{\mathrm{F}}^{(11)}(k) r_\mu S_{\mathrm{F}}^{(11)}(k+p_2) r_\sigma S_{\mathrm{F}}^{(11)}(k+p_1+p_2) r_\nu S_{\mathrm{F}}^{(11)} \times (k+p_1+p_2+p_3) r_\rho \tag{2.15}$$

I_c(figure 2(c))

$$= \mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} S_{\mathrm{F}}^{(11)}(k) r_\mu S_{\mathrm{F}}^{(11)}(k+p_2) r_\nu S_{\mathrm{F}}^{(11)}(k+p_2+p_3) r_\sigma S_{\mathrm{F}}^{(11)} \times (k+p_1+p_2+p_3) r_\rho . \tag{2.16}$$

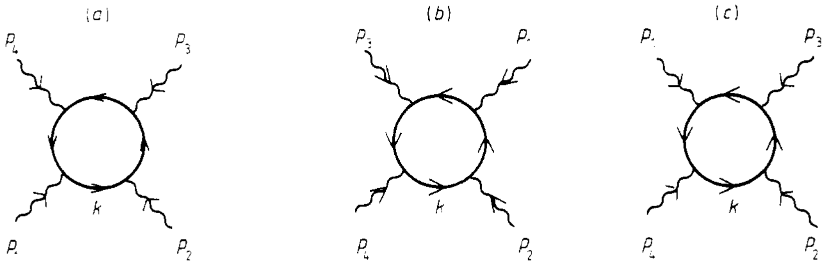

Figure 2. Photon-photon scattering diagram.

We multiply $P_{1\sigma}$ on $I_a(b, c)$. The sum is expected to vanish due to gauge invariance. By the use of Takahashi's identity, (2.9), one finds

$$P_{1\sigma} I_a = \mathrm{i}\,\mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} [S_{\mathrm{F}}(k) r_\mu S_{\mathrm{F}}(k+p_1+p_2) r_\nu S_{\mathrm{F}}(k+p_1+p_2+p_3) r_\rho - S_{\mathrm{F}}(k) r_\mu S_{\mathrm{F}}(k+p_2) r_\nu S_{\mathrm{F}}(k+p_2+p_3) r_\rho] \tag{2.17}$$

$$P_{1\sigma} I_b = \mathrm{i}\,\mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} [S_{\mathrm{F}}(k) r_\mu S_{\mathrm{F}}(k+p_2) r_\nu S_{\mathrm{F}}(k+p_1+p_2+p_3) r_\rho - S_{\mathrm{F}}(k) r_\mu S_{\mathrm{F}}(k+p_1+p_2) r_\nu S_{\mathrm{F}}(k+p_1+p_2+p_3) r_\rho] \tag{2.18}$$

$$P_{1\sigma} I_c = \mathrm{i}\,\mathrm{Tr} \int \frac{\mathrm{d}^n k}{(2\pi)^n} [S_{\mathrm{F}}(k) r_\mu S_{\mathrm{F}}(k+p_2) r_\nu S_{\mathrm{F}}(k+p_2+p_3) r_\rho - S_{\mathrm{F}}(k) r_\mu S_{\mathrm{F}}(k+p_2) r_\nu S_{\mathrm{F}}(k+p_1+p_2+p_3) r_\rho]. \tag{2.19}$$

Thus they add up to be zero.

2.5. $P_\mu \pi_{\mu\nu}(p) = 0$ (two-loop)

At two-loop level one must consider a new type of diagrams which are absent at $T = 0$. They (figures 4(b), 5(b)) appear due to the fact that ψ and $\tilde{\psi}$ (A_μ and $\tilde{A}_\mu$) mix in the thermo propagator. In figures 3, 4 and 5, $\langle 1 \rangle$, $\langle 2 \rangle$ and $\langle 12 \rangle$ denote respectively (11),

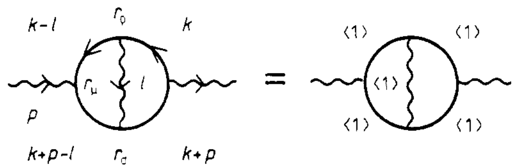

Figure 3. Two-loop photon self-energy diagrams.

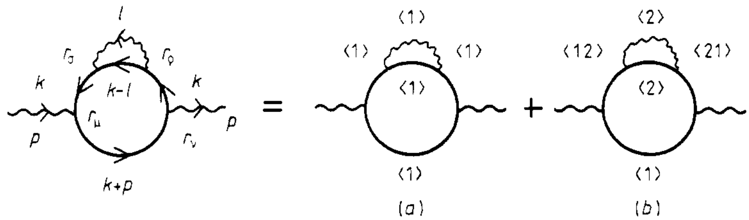

Figure 4. As figure 3.

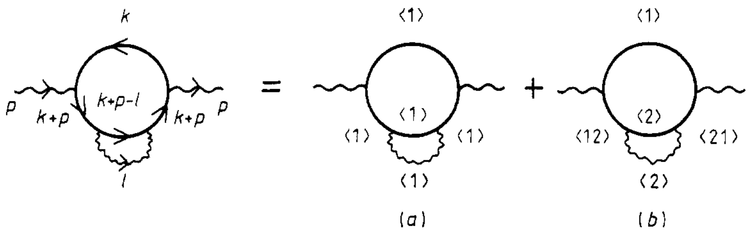

Figure 5. As figure 3.

(22) and (12) elements of fermion or photon thermo propagator. The photon propagator is given by Matsumoto *et al* [10],

$$D_{\mu\nu}(k) \equiv \begin{vmatrix} \langle 1\rangle & \langle 12\rangle \\ \langle 21\rangle & \langle 2\rangle \end{vmatrix}$$
$$= -\left(g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right) \times \begin{vmatrix} D_0^1(k) + 2\pi\delta(k^2)f_{\mathrm{B}} & 2\pi\delta(k^2)g_{\mathrm{B}} \\ 2\pi\delta(k^2)g_{\mathrm{B}} & D_0^2(k) + 2\pi\delta(k^2)f_{\mathrm{B}} \end{vmatrix} \tag{2.20}$$

where

$$D_0^1(k) = \mathrm{i}/(k^2 + \mathrm{i}\varepsilon)^{-1} \qquad D_0^2(k) = -\mathrm{i}/(k^2 - \mathrm{i}\varepsilon)^{-1}. \tag{2.21}$$

$$I_3(\text{figure } 3) = \mathrm{Tr}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\, r_\sigma S_{\mathrm{F}}^{(11)}(p+k) r_\nu S_{\mathrm{F}}^{(11)}(k) r_\rho S_{\mathrm{F}}^{(11)}$$
$$\times (k-l) \not{P} S_{\mathrm{F}}^{(11)}(k-l+p) D_{\rho\sigma}^{(11)}(l). \tag{2.22}$$

In this integral one can use the identity (2.9), and we obtain

$$I_3 = \mathrm{i}\,\mathrm{Tr}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\, r_\sigma S_{\mathrm{F}}^{(11)}(p+k) r_\nu S_{\mathrm{F}}^{(11)}(k) r_\rho S_{\mathrm{F}}^{(11)}(k-l) D_{\rho\sigma}^{(11)}(l) \tag{2.23a}$$

$$-\mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,r_\sigma S_{\mathrm{F}}^{(11)}(p+k)r_\nu S_{\mathrm{F}}^{(11)}(k)r_\rho S_{\mathrm{F}}^{(11)}$$
$$\times(k-l+p)D_{\rho\sigma}^{(11)}(l). \tag{2.23b}$$

I_{4a}(figure 4(a)) and I_{4b}(figure 4(b)) must be treated together since each integral is mathematically ill defined due to the $\delta^2(k^2-m^2)$ term. Fortunately in TFD two terms combine to become well defined [9, 13]. Leaving the detail of calculation in the appendix the results are

$$I_{4a}+I_{4b}=\mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,r_\rho S_{\mathrm{F}}^{(11)}(k-l)r_\sigma(\not{k}+m)r_\nu(\not{k}+m)D_{\rho\sigma}(l)Q(k) \tag{2.24a}$$

$$-\mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,S_{\mathrm{F}}^{(11)}(k+p)r_\nu S_{\mathrm{F}}^{(11)}(k)r_\rho S_{\mathrm{F}}^{(11)}(k-l)r_\sigma D_{\rho\sigma}^{(11)}(l). \tag{2.24b}$$

Likewise I_{5a} (figure 5(a)) and I_{5b} (figure 5(b)) turn out to be

$$I_{5a}+I_{5b}=-\mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,(\not{k}+\not{p}+m)r_\nu(\not{k}+\not{p}+m)r_\rho$$
$$\times S_{\mathrm{F}}^{(11)}(k-l+p)r_\sigma D_{\rho\sigma}(l)Q(k+p) \tag{2.25a}$$
$$+\mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,S_{\mathrm{F}}^{(11)}(k+p)r_\nu S_{\mathrm{F}}^{(11)}(k)r_\rho$$
$$\times S_{\mathrm{F}}^{(11)}(k-l+p)r_\sigma D_{\rho\sigma}^{(11)}(l). \tag{2.25b}$$

(2.24b)((2.25b)) cancels with (2.23a)((2.23b)) and (2.25a) cancels with (2.24a) after a change of variable $k+p\to k$.

We have confirmed that gauge invariance expressed as $P_\mu\pi_{\mu\nu}(p)=0$ is respected in TFD perturbation up to two-loop level. However there is a subtle technical point involved in our verification. We would like to explain it below.

$(k^2-m^2)Q(k)$ of (A8) can lead to three different results:

(1)
$$\begin{aligned}(k^2-m^2)Q(k)&=(k^2-m^2)\{(\Delta_0^1(k))^2-[(\Delta_0^1(k))^2-(\Delta_0^2(k))^2]f_{\mathrm{F}}(k)\}\\&=\mathrm{i}[\Delta_0^1(k)-(\Delta_0^1(k)-\Delta_0^2(k))f_{\mathrm{F}}(k)]\\&=\mathrm{i}(\Delta_0^1(k)-2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k))\end{aligned} \tag{2.26}$$

where we have set

$$(k^2-m^2)(\Delta_0^1(k))^2=\mathrm{i}\Delta_0^1(k) \qquad (k^2-m^2)(\Delta_0^2(k))^2=\mathrm{i}\Delta_0^2(k) \tag{2.27a, b}$$

(2)
$$\begin{aligned}(k^2-m^2)Q(t)&=(k^2-m^2)\left((\Delta_0^1(k))^2-2P\frac{\mathrm{i}}{k^2-m^2}2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)\right)\\&=\mathrm{i}\Delta_0^1(k)-\left((k^2-m^2)2P\frac{\mathrm{i}}{k^2-m^2}\right)2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)\\&=\mathrm{i}(\Delta_0^1(k)-4\pi\delta(k^2-m^2)f_{\mathrm{F}}(k))\end{aligned} \tag{2.28}$$

(3)
$$\begin{aligned}(k^2-m^2)Q(k)&=\mathrm{i}\Delta_0^1(k)-2P\frac{1}{k^2-m^2}[(k^2-m^2)2\pi\delta(k^2-m^2)]f_{\mathrm{F}}(k)\\&=\mathrm{i}\Delta_0^1(k).\end{aligned} \tag{2.29}$$

Results (2) and (3) lead to the breaking of gauge invariance. We have chosen (1). Our prescription for the product of distributions reads in a generalised form as,

$$(k^2-m^2)^N[(\Delta_0^1(k))^M-(\Delta_0^2(k))^M]=\mathrm{i}^{M-N}[(\Delta_0^1(k))^{M-N}-(\Delta_0^2(k))^{M-N}] \tag{2.30}$$

and it is in accordance with the one used in the $T=0$ perturbation. Further, in carrying out an integral which contains a factor of (2.30) one may use a mass derivative formula,

$$(1/N!)(\mathrm{i}(\partial/\partial m^2))^N 2\pi\delta(k^2-m^2)=(\Delta_0^1(k))^{N+1}-(\Delta_0^2(k))^{N+1}. \tag{2.31}$$

We have independently checked gauge invariance in the imaginary time formalism (ITF) [11, 14]. The ITF version of Takahashi's identity (2.9) which one obtains by the replacement of k_0 with $\mathrm{i}(2n+1)\pi/\beta$ holds true without a subsidiary condition like (2.8). Therefore one may freely make use of it at the higher orders. Then one can easily confirm the gauge invariance at two-loop level.

In the present paper we provided a proof of gauge invariance in QED at two-loop level. Earlier the Goldstone theorem at $T\neq 0$ [6] was investigated and it is found that Nambu-Goldstone mode remains massless at one-loop level [15]. So far we have not found an example where the WT identity is broken in finite temperature perturbation except (2.10).

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Appendix

In this appendix we give the derivation of (2.24a, b) and (2.25a, b). In figures 4 and 5, we cannot make direct use of Takahashi's identity due to the reason explained in the paragraph after equation (2.9).

$$I_{4a}=\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,r_\nu(\not k+m)r_\rho S_\mathrm{F}^{(11)}(k-l)r_\sigma(\not k+m)\not P(\not k+\not p+m)D_{\rho\sigma}^{(11)}(l)$$
$$\times(\Delta_0^1(k)-2\pi\delta(k^2-m^2)f_\mathrm{F}(k))^2[\Delta_0^1(k+p)-2\pi\delta((k+p)^2-m^2)f_\mathrm{F}(k+p)] \tag{A1}$$

$$I_{4b}=\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,r_\nu(\not k+m)r_\rho S_\mathrm{F}^{(11)}(k-l)r_\sigma(\not k+m)\not P(\not k+\not p+m)D_{\rho\sigma}^{(11)}(l)$$
$$\times(2\pi\delta(k^2-m^2)g_\mathrm{F}(k))^2[\Delta_0^1(k+p)-2\pi\delta((k+p)^2-m^2)f_\mathrm{F}(k+p)]. \tag{A2}$$

In (A1) we employed a formula

$$\int\mathrm{d}^n l\, S_\mathrm{F}^{(22)}(k-l)D^{(22)}(l)=\int\mathrm{d}^n l\, S_\mathrm{F}^{(11)}(k-l)D^{(11)}(l) \tag{A3}$$

$$I_{4a}+I_{4b}=\mathrm{Tr}\int\frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n}\,r_\nu(\not k+m)r_\rho S_\mathrm{F}^{(11)}(k-l)r_\sigma(\not k+m)\not P(\not k+\not p+m)D_{\rho\sigma}^{(11)}(l)$$

$$\times\left((\Delta_0^1(k))^2 - 2P\frac{\mathrm{i}}{k^2-m^2}2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)\right)$$
$$\times[\Delta_0^1(k+p) - 2\pi\delta((k+p)^2-m^2)f_{\mathrm{F}}(k+p)]. \tag{A4}$$

(A4) is derived in the following manner:

$$\begin{aligned}
&[\Delta_0^1(k) - 2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)]^2 + [2\pi\delta(k^2-m^2)g_{\mathrm{F}}(k)]^2 \\
&\quad = (\Delta_2^1(k))^2 - 2\Delta_0^1(k)(\Delta_0^1(k)-\Delta_0^2(k))f_{\mathrm{F}}(k) + (\Delta_0^1(k)-\Delta_0^2(k))^2(f_{\mathrm{F}}^2(k)+g_{\mathrm{F}}^2(k)) \\
&\quad = (\Delta_0^1(k))^2 - [(\Delta_0^1(k))^2 - (\Delta_0^2(k))^2]f_{\mathrm{F}}(k) \\
&\quad = (\Delta_0^1(k))^2 - (\Delta_0^1(k)+\Delta_0^2(k))(\Delta_0^1(k)-\Delta_0^2(k))f_{\mathrm{F}}(k) \\
&\quad = (\Delta_0^1(k))^2 - 2P\frac{\mathrm{i}}{k^2-m^2}2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k) \qquad (P\text{: principal part}).
\end{aligned} \tag{A5}$$

Note that the following identity holds

$$\begin{aligned}
(\not{k}+m)\not{P}(\not{k}+\not{p}+m) &= -(\not{k}+m)[\not{k}-m-(\not{k}+\not{p}-m)](\not{k}+\not{p}+m) \\
&= -(k^2-m^2)(\not{k}+\not{p}+m) + (\not{k}+m)[(k+p)^2-m^2].
\end{aligned} \tag{A6}$$

Then by the use of (A6) we find

$$I_{4a}+I_{4b} = \mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^nk}{(2\pi)^n}\frac{d^nl}{(2\pi)^n}r_\nu(\not{k}+m)r_\sigma S_{\mathrm{F}}^{(11)}(k-l)r_\sigma(\not{k}+m)D_{\rho\sigma}^{(11)}(l)\times Q(k) \tag{A7a}$$

$$\left[Q(k) \equiv \left((\Delta_0^1(k))^2 - 2\mathrm{P}\frac{\mathrm{i}}{k^2-m^2}2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)\right)\right]$$

$$-\mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^nk}{(2\pi)^n}\frac{\mathrm{d}^nl}{(2\pi)^n}r_\nu(\not{k}+m)r_\rho S_{\mathrm{F}}^{(11)}(k-l)r_\sigma S_{\mathrm{F}}^{(11)}$$
$$\times(k+p)D_{\rho\sigma}^{(11)}(l)(k^2-m^2)Q(k). \tag{A7b}$$

Here we set

$$(k^2-m^2)Q(k) = \mathrm{i}(\Delta_0^1(k) - 2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)). \tag{A8}$$

Then

$$(\mathrm{A}7b) = -\mathrm{i}\,\mathrm{Tr}\int\frac{\mathrm{d}^nk}{(2\pi)^n}\frac{\mathrm{d}^nl}{(2\pi)^n}r_\nu S_{\mathrm{F}}^{(11)}(k)r_\rho S_{\mathrm{F}}^{(11)}(k-l)r_\sigma S_{\mathrm{F}}^{(11)}(k+p)D_{\rho\sigma}^{(11)}(l). \tag{A9}$$

(A7a) and (A9) are respectively (2.24a) and (2.24b) in the text.

In an analogous manner

$$I_{5a} = \mathrm{Tr}\int\frac{\mathrm{d}^nk}{(2\pi)^n}\frac{\mathrm{d}^nl}{(2\pi)^n}r_\nu(\not{k}+m)\not{P}(\not{k}+\not{p}+m)r_\rho S_{\mathrm{F}}^{(11)}(k-l+p)r_\sigma(\not{k}+\not{p}+m)D_{\rho\sigma}^{(11)}(l)$$
$$\times[\Delta_0^1(k+p) - 2\pi\delta((k+p)^2-m^2)f_{\mathrm{F}}(k)]^2(\Delta_0^1(k) - 2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)) \tag{A10}$$

$$I_{5b} = \mathrm{Tr}\int\frac{\mathrm{d}^nk}{(2\pi)^n}\frac{\mathrm{d}^nl}{(2\pi)^n}r_\nu(\not{k}+m)\not{P}(\not{k}+\not{p}+m)r_\rho S_{\mathrm{F}}^{(11)}(k-l+p)r_\sigma(\not{k}+\not{p}+m)D_{\rho\sigma}^{(11)}(l)$$
$$\times(2\pi\delta(k+p)^2-m^2)g_{\mathrm{F}}(k))^2(\Delta_0^1(k) - 2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)) \tag{A11}$$

$$I_{5a}+I_{5b}=\mathrm{Tr}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n} r_\nu(\not{k}+m)\not{P}(\not{k}+\not{p}+m)r_\rho S_{\mathrm{F}}^{(11)}$$
$$\times(k-l+p)r_\sigma(\not{k}+\not{p}+m)D_{\rho\sigma}^{(11)}(l)$$
$$\times\left(\Delta_0^1(k+p)^2-2\mathrm{P}\frac{\mathrm{i}}{k+p-m^2}2\pi\delta(k+p)^2-m^2\right)$$
$$\times[\Delta_0^1(k)-2\pi\delta(k^2-m^2)f_{\mathrm{F}}(k)]. \qquad \text{(A12)}$$

Using (A6) one arrives at

$$I_{5a}+I_{5b}=-\mathrm{i}\,\mathrm{Tr}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n} r_\nu(\not{k}+\not{p}+m)r_\rho S_{\mathrm{F}}^{(11)}(k-l+p)r_\sigma$$
$$\times(\not{k}+\not{p}+m)D_{\rho\sigma}^{(11)}(l)Q(k+p) \qquad \text{(A13}a\text{)}$$
$$+\mathrm{i}\,\mathrm{Tr}\int \frac{\mathrm{d}^n k}{(2\pi)^n}\frac{\mathrm{d}^n l}{(2\pi)^n} r_\nu S_{\mathrm{F}}^{(11)}(k)r_\rho S_{\mathrm{F}}^{(11)}(k-l+p)r_\sigma S_{\mathrm{F}}^{(11)}(k+p)D_{\rho\sigma}^{(11)}(l). \qquad \text{(A13}b\text{)}$$

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